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Extension of a new approach towards accurate stress analysis of laminates subjected to thermomechanical loading

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Abstract The stress-analysis method presented in this paper is based on the extension of a successful, two-dimensional generalised laminate theory towards consideration of thermo-mechanical deformation effects. Like its conventional counterparts, that generalised equivalent-single-layer (ESL) laminate model makes use of a fixed and small number of unknowns. Its success is, however, based on the incorporation into its approximate, thermo-mechanical displacement field of the exact elasticity distributions available for corresponding simply supported structural components. With this incorporation, the ESL laminate model acquires benefits of a corresponding layer-wise theory. Most importantly, the proposed ESL theory and three-dimensional thermo-elasticity yield identical stress distributions for simply supported laminates. For other sets of edge boundary conditions, solutions and therefore detailed response characteristics that are based on such an ESL theory are accurate away of the laminate edges. If necessary, accuracy of detailed response characteristics can be further improved on a predictor–corrector basis.

Keywords Composite laminates · Mathematical modelling · Stress analysis · Thermo-elasticity

1 Introduction

The approximately two-hundred-years-long history of the theory of linear elasticity, along with associated research areas and topics, contains paradoxes, points of controversy as well as strong relevant debates, with the latter having contributed substantially to the progress and advance of the subject. The bending problem of a thin elastic rectangular plate made of isotropic material, having its edges clamped and being subjected to uniform pressure is mentioned, in particular, as the first among several examples that could fit the purposes of the present paper. In mathematical terms, that boundary-value problem is described as follows:

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$$D\nabla^2\nabla^2 w = q(x,y), \quad w|_{x=0,L_x} = \left.\frac{\partial w}{\partial x}\right|_{x=0,L_x} = 0, \;\; w\bigg|_{y=0,L_y} = \left.\frac{\partial w}{\partial y}\right|_{y=0,L_y} = 0, \tag{1}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the form of the Laplace operator in Cartesian co-ordinates x and y, the constants L_x and L_y represent the length and the width of the plate, respectively, and hence define the domain of the problem, D is the flexural rigidity of the plate and q represents the applied pressure. Extensive accounts of solutions relevant to this problem, proposed, attempted and achieved by many well-known scientists, mathematicians and engineers, are presented in [1]. Reference [1] also comments on the reasons that this problem has attracted the interest of both mathematicians and engineers; it gives detailed information and invaluable further sources of the fascinating (as refers to it) long history of that classical problem and, in this context, it even mentions of the manner that the (now famous) relevant study by Ritz [2] was initially misplaced or mistreated. Moreover, a comment placed early in the Introduction of [1], according to which "the mathematician is interested in the problem for its own sake and not for its practical application, whereas the engineer is interested in the practical problem, and he uses his mathematics merely as a tool", can be generalised and furnish to reflect the fact that theoretical and applied sciences complement and assist one another.

In this context it is instructive to note that (1) represents only one of the many boundary-value problems stemming from applications of the classical plate theory (CPT); namely, a two-dimensional mathematical model which, although substantially inferior when compared with the three-dimensional theory of linear isotropic elasticity counterpart, was until recently, and perhaps still is, a successful principal design tool in engineering and the applied sciences. Associated with Kirchhoff's name [3], the static version of the CPT and the order (four) of the resulting partial differential equation (PDE) necessitates application of two boundary conditions along each edge of the plate; the pair of boundary conditions appearing in (1) is only a representative sample of other possible pairs of edge boundary conditions. As Reissner described [4], "What makes Kirchhoff's result concerning the boundary conditions remarkable is that "physical intuition" leads one to expect *three* natural conditions, such as the three conditions of prescribed edge-force intensity, bending-moment intensity, and twisting-moment intensity, as had earlier been demanded by no lesser an authority than the great Poisson". As it is further detailed in [4], Reissner remembered "clearly that he was much intrigued by this boundary-condition *paradox*, while learning about plates in a course on Statik der Baukonstruktionen which was at that time offered by his father at the Technische Hochschule Berlin. It was clear that, given the reality of three edge conditions, a sixth-order rather than a fourth-order differential equation ought to be in charge of the problem". Becoming later the first who eventually obtained the anticipated sixth-order differential equation, about a decade after his graduation from Technische Hochschule Berlin, Reissner added in 1985 [4]: "Today in retrospect, one feels that the mere raising of the question should have led to its straightforward resolution. In contrast to this, the question had remained unresolved for more than 80 years, and would remain so for 10 more years. Once an answer had been found of how a rational sixth-order differential equation formulation would come out [5,6], somewhat less simply than indicated above, it also became apparent why the "paradox" arose in the first place, and alternative ways of resolving it were developed in short order".

The plate theory presented in [5,6] and those that followed it have offered substantial improvements of the CPT considerations, at both theoretical and practical level, and have therefore extended considerably the range of applicability of two-dimensional plate modelling. However, by nature, two-dimensional modelling of thin-walled structures can never become as accurate as three-dimensional theory of linear elasticity. As already mentioned, the latter provides the basic mathematical models and tools for theoretical study and prediction of the behaviour of common structural components met in engineering and applied sciences for approximately two hundred years. Although linear in form, the resulting complicated, high-order PDEs were initially solved for relatively simple problems only. For many problems that the theory is invited to model advanced practical situations, as happens in modern mechanics of composite and fibre-reinforced structural components, those PDEs had to wait for much more than a century, until the evolution of high-speed computational means made more widely understood the theoretical and practical benefits stemming from their solution. Nevertheless, it is still the minimisation of energy functionals (e.g. the Ritz method), rather than methods directly applied on equivalent three-dimensional elasticity PDEs which is regarded as computationally feasible, almost always provide the best possible and most accurate relevant information sought.

Although not much is now considered as computationally impossible in structural analysis, application of energy-minimisation approaches is still less economical if compared with the results of possible analytical methods that might be applied directly to equivalent equilibrium PDEs, particularly when someone deals with the mechanics of multilayered, highly anisotropic thin-walled components on the basis of three-dimensional elasticity considerations. The through-thickness continuity constraints that have to be applied to both displacements and inter-laminar stresses may be referred to as an adequate justification of the latter argument. Three-dimensional elasticity PDEs may, however, be solved exactly only in particular cases and mainly for problems referring to structural components having all their edges simply supported. Even in such cases though, the possible appearance of variable coefficients, which are mainly but not exclusively associated with consideration of curvature effects (shell-type structures), may impose considerable additional difficulty when attempting to find the solution of the involved PDEs.

The latter arguments justify completely the enormous amount of effort, work and time spent by many bright researchers who, starting with the pre-elasticity era searches of the Bernoullis and Euler, keep developing and using relevant one- and two-dimensional mathematical models that predict as successfully as possible the behaviour of thin-walled, three-dimensional structural components. As already detailed, such lower-dimensional modelling has, of course, limitations and has therefore both received and survived criticisms from several scholars, some of whom have also contributed considerably to its development (e.g. [7,8]). The relatively recent interest in composite, fibre-reinforced materials and "smart" structural components gave a boost to the subject which, departing from its early developments (e.g. [3–6]) and combined with the parallel evolution of computational means, has grown considerably over the last 35 years, and will continue to do so (for more recent developments, see for instance [9,10]). In this connection, some advanced kind of elastic-plate modelling proposed in [9,10] has produced relevant, two-dimensional mathematical models that predict displacement, strain and stress distributions identical to those predicted by the three-dimensional theory of elasticity for elastic plates having simply supported edges. This is a remarkable output for a two-dimensional mathematical model and can evidently serve as a point of reference towards future developments in the subject. Moreover, it gives rise to a new accurate stress-analysis method, applications of which are so far confined within the relatively simple geometrical features of flat plates and plate-like composite laminates (e.g. [10–16]).

The possibility for further expansion of the method towards accurate stress analysis of relevant curved structural components and component assemblies is, however, also anticipated in [10]. There, the manner is further detailed for the construction of relatively simple computer subroutines containing solutions of three-dimensional elasticity PDEs that represent appropriate cylindrical-shell-components modelling. Although not detailed in [10], a subsequent merger of such type of three-dimensional elasticity solutions with the appropriate two-dimensional generalised cylindrical shell model would serve directly towards the new method's expansion within the bounds of purely mechanical response of curved structural components. Expansion of the method's applicability to areas and disciplines of associated research interest is, however, also desirable and possible. For instance, the current interest in developing so-called smart materials and structures, as well as the associated interest in studying the behaviour and exploiting the properties of relevant structural components, directs possible developments and the method's expansion towards the fields of thermo-elasticity and piezoelectricity. Under these considerations, this paper is considered as an initial step towards the extension of the outlined new method in the field of thermo-elastic stress analysis of structural components and, as such, it essentially follows the pattern employed in [10] for corresponding structural components that exhibit purely mechanical response.

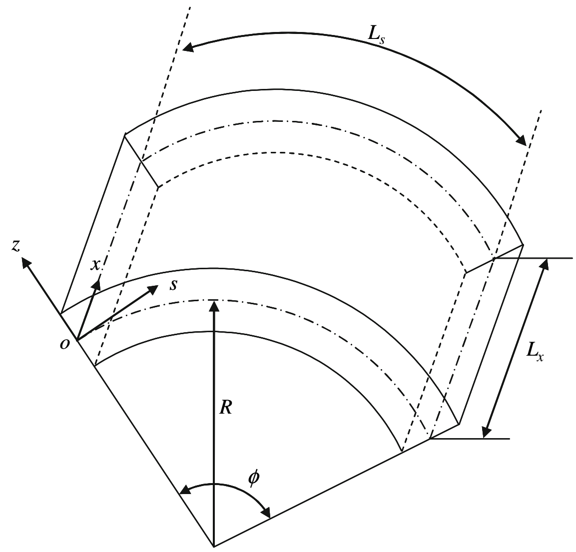


Fig. 1 Laminated cylinder geometry and coordinate system

It should therefore be emphasised that this paper is inevitably briefer than [10]. Where appropriate, it accordingly outlines possible future developments by citing relevant references than through detailed and full expansion of already existing relevant work. In some detail though, the proposed accurate thermo-elasticity stress-analysis method will still be based on a generalised equivalent-single-layer (ESL) laminate model that makes use of a fixed and small number of unknowns (degrees of freedom). The method's success emerges again from the appropriate incorporation of the displacement field of exact thermo-elasticity distributions available for corresponding simply supported structural components (Sect. 2) into the approximate displacement field of an ESL laminate model (Sects. 3–5). With this incorporation, the ESL laminate model acquires benefits of a corresponding layer-wise theory. As already mentioned, the most important feature of this merger is that the proposed ESL theory and three-dimensional thermo-elasticity yield identical stress distributions for simply supported laminates. For other sets of edge boundary conditions, solutions and therefore detailed response characteristics predicted through the solution of the governing PDEs of such an ESL theory are accurate everywhere apart from a neighbourhood of the laminate edges (Sect. 6). If necessary, the accuracy of detailed response characteristics can further be improved with the use of a predictor–corrector approach.

2 Three-dimensional thermo-elasticity solutions for laminated composite plates and cylinders

Consider a circular cylindrical panel with constant thickness h and axial length L_x (Fig. 1). The radius and the circumferential length of its middle surface are denoted by R and L_s, respectively, so $\phi = L_s/R$ represents its shallowness angle; upon choosing $\phi = 0$ or $\phi = 2\pi$, one obtains the geometry of the flat plate or completely circular cylinder, respectively, as particular cases. The axial, circumferential and normal-to the middle-surface coordinate-length parameters are denoted by x, s and z, respectively. For relative simplicity, the cylinder is assumed to be made of a linearly elastic orthotropic material whose principle axes of orthotropy coincide with the axes of the curvilinear coordinate system employed.

Any type of thermo-elasticity theory can be considered and would be equally good in serving the general purposes of this section, including the conventional thermo-elasticity theory [17] as well as relevant theories with second sound (e.g. [18]). For the purposes of the present section, the generalised thermo-elasticity theory with second sound due to Kaliski [19], Dhaliwal and Sherief [20] and Li [21] is employed as a reference theory. Relevant results obtained by using conventional thermo-elasticity theory can subsequently be obtained as a particular case; these will be employed later in an example application. Under these considerations and upon neglecting, for simplicity, the effects of heat sources, the stress equations of motion may be brought into the following form:

$$\begin{aligned}
&\sigma_{xx,x} + \tau_{xs,s} + \tau_{xz,z} + R^{-1}\left(1+\frac{z}{R}\right)^{-1}\tau_{xz} = \rho U_{,tt},\\
&\tau_{sx,x} + \sigma_{ss,s} + \tau_{sz,z} + 2R^{-1}\left(1+\frac{z}{R}\right)^{-1}\tau_{sz} = \rho V_{,tt},\\
&\tau_{zx,x} + \tau_{zs,s} + \sigma_{zz,z} + R^{-1}\left(1+\frac{z}{R}\right)^{-1}(\sigma_{zz}-\sigma_{ss}) = \rho W_{,tt},\\
&k_z\left[T_{,zz} + R^{-1}\left(1+\frac{z}{R}\right)^{-1}T_{,z}\right] + k_s T_{,ss} + k_x T_{,xx}\\
&\quad = \left(1+\tau_0\frac{\partial}{\partial t}\right)\left\{\rho c T_{,t} + T_0\left[b_z W_{,zt} + b_s\left(V_{,st} + R^{-1}\left(1+\frac{z}{R}\right)^{-1}W_{,t}\right) + b_z W_{,xt}\right]\right\},
\end{aligned} \tag{2}$$

where U, V and W are the displacement components along the axial, circumferential and radial directions, respectively, and a comma denotes partial differentiation with respect to the indicated variable(s). Moreover, T_0 and T are the initial temperature and the absolute temperature, respectively, and t denotes time. Finally, ρ is the material density, k_i and b_i are the non-zero components of thermal conductivity and thermo-elasticity tensor, respectively, c denotes the specific heat per unit mass and τ_0 is the thermal relaxation parameter.

For infinitesimal deformations, the Duhamel–Neumann thermo-elasticity constitutive equations can be expressed as follows:

$$\begin{Bmatrix}\sigma_x\\ \sigma_s\\ \sigma_z\\ \tau_{sz}\\ \tau_{xz}\\ \tau_{xs}\end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x-\alpha_x T\\ \varepsilon_s-\alpha_s T\\ \varepsilon_z-\alpha_z T\\ \gamma_{sz}\\ \gamma_{xz}\\ \gamma_{xs}\end{Bmatrix}, \tag{3}$$

where C_{ij} $(i,j = 1,2,\ldots,6)$ are the elastic moduli of an orthotropic material and α_i $(i = x,y,z)$ are the coefficients of linear thermal expansion. The following kinematic relations,

$$\begin{aligned}
&\varepsilon_x = U_{,x}, \quad \varepsilon_s = V_{,s} + R^{-1}\left(1+\frac{z}{R}\right)^{-1}W, \quad \varepsilon_z = W_{,z},\\
&\gamma_{sz} = W_{,s} + V_{,z} - R^{-1}\left(1+\frac{z}{R}\right)^{-1}V, \quad \gamma_{xz} = W_{,x} + U_{,z}, \quad \gamma_{xs} = V_{,x} + U_{,s},
\end{aligned} \tag{4}$$

complement the constitutive equations (3) and reveal that, due to the appearance of terms involving $(1 + z/R)^{-1}$, their introduction into (2) yields a system of four, Navier-type, simultaneous PDEs with variable coefficients. The latter are differential equations obtained in terms of the unknown displacement components and temperature; their explicit forms can be found in [22].

For cylindrical panels having all four of their edges simply supported, the obtained set of simultaneous PDEs admits a separable solution of the form,

$$\begin{aligned}
&U(x,s,z;t) = \Phi_1(z)\cos\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi s}{L_s}\right)\Omega(t), \quad V(x,s,z;t) = \Phi_2(z)\sin\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi s}{L_s}\right)\Omega(t),\\
&W(x,s,z;t) = \Psi(z)\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi s}{L_s}\right)\Omega(t), \quad T(x,s,z,t) = \Theta(z)\sin\left(\frac{m\pi x}{L_x}\right)\sin\left(\frac{n\pi s}{L_s}\right)\Omega(t).
\end{aligned} \tag{5}$$

Here, m and n are integers representing axial and circumferential wave parameters, respectively, while Φ_1, Φ_2, Ψ and Θ are unknown functions of the transverse co-ordinate parameter, z. In the case of free vibrations, the function $\Omega(t)$ is chosen to be periodic in time while in static problems it is replaced by 1. In either case, introduction of (4) into the aforementioned "Navier-type" PDEs [22] eliminates dependency on the in-plane co-ordinates, x and s. A system of four simultaneous ordinary differential equations (ODEs) is thus obtained. It possesses variable coefficients and its solution yields the form of the four unknown functions Φ_1, Φ_2, Ψ and Θ. These functions represent the exact through-thickness distribution of the displacement components and the temperature, while corresponding, exact through-thickness strain and stress distributions are obtained upon subsequent use of Eqs. (4) and (3), respectively.

It should be noted at this point that, in the particular case of flat plate-like structural components ($R = \infty$) the latter set of ODEs possesses constant coefficients and can therefore be solved exactly with standard methods of differential calculus. It follows that the functions Φ_1, Φ_2, Ψ and Θ can be finally expressed in terms of elementary mathematical functions. They can therefore be converted easily into the form of computer subroutines and stored into a computer's memory for later possible use. As will be seen in the subsequent sections devoted to a particular flat-plate application, this kind of computerised implementation of Φ_1, Φ_2, Ψ and Θ is of profound importance for their merger with a generalised two-dimensional (2D) plate theory and, hence, for the efficient computational implementation of the proposed accurate stress-analysis method.

In the case of shell-type structural components, the difficulty of the variable coefficients appearing in (2), (4) and, hence, in the corresponding Navier-type differential equations is confronted, with relative ease, by means of the successive approximation method introduced in [23] and used later in a series of relevant publications (e.g. [24–29]). Notably, that successive approximation method has shown rigorously [30] to be equivalent to any exact analytical method available for the solution of relevant ODEs with variable coefficients. In the particular case of stationary thermo-elastic analysis of cylinders and cylindrical panels, detailed accounts of its efficiency, as well as easy implementation in terms of 6×6 matrix operations, are given in [28]. In the latter case, the heat-conduction equation (2.d) becomes uncoupled from the static equilibrium version of Eqs. (2.a-c) and the method's computer implementation resembles its purely mechanical counterpart (e.g. [10]). However, regardless of the type of the particular thermo-elastic problem considered, the computer implementation of the functions Φ_1, Φ_2, Ψ and Θ will be again of profound importance for their merger with some appropriate generalised 2D shell theory (e.g. [31,32]). It is emphasised that, in the case of composite laminates, each individual layer is treated as a materially homogeneous structural component and the final, composite form of Φ_1, Φ_2, Ψ and Θ is still obtained by using 6×6 matrix operations, upon requiring continuity of displacements and inter-laminar stresses at the laminate material interfaces (e.g. [28]).

3 Outline of a 2D generalised six-degree-of-freedom plate theory for use in stationary thermo-elastic analysis

In stationary thermo-elastic applications, the heat-conduction equation is uncoupled from the static remaining equilibrium equations and can therefore be solved independently of the latter. It follows that the principal governing equations of the 2D generalised, six-degree-of-freedom plate theory (G6DOFPT) [10,11] are still adequate for the purposes of the present section and they are quoted next for self-sufficiency. It is recalled that $R = \infty$ and, therefore, $\phi = 0$ in the flat-plate case considered in this section (Fig. 1). Moreover, replacement of the parameters s and L_s with y and L_y, respectively, will conveniently reflect the fact that a Cartesian rather than a curvilinear co-ordinate-system description is employed in what follows.

Assume that, in general, the plate is made of some transversely inhomogeneous, linearly elastic anisotropic material; the laminated composite plate is thus considered as a particular case by assuming through-thickness piece-wise constant inhomogeneity. Development of G6DOFPT begins with the displacement field approximation:

$$\begin{aligned} U(x,y,z) &= u_0(x,y) - z\, w_{0,x}(x,y) + \varphi_1(z)\, u_1(x,y), \\ V(x,y,z) &= v_0(x,y) - z\, w_{0,y}(x,y) + \varphi_2(z)\, v_1(x,y), \\ W(x,y,z) &= w_0(x,y) + \psi(z)\, w_1(x,y), \end{aligned} \tag{6}$$

where u_0, v_0 and w_0 represent the unknown displacements of the plate middle plane. These and u_1, v_1 and w_1, which represent the unknown values of the transverse strains on the plate middle plane, are the six main unknowns (degrees of freedom) of the theory. The functions $\varphi_1(z)$, $\varphi_2(z)$ and $\psi(z)$ are assumed to be given functions of the transverse co-ordinate parameter that can dictate the shape of transverse shear and normal strains. The latter may be chosen in any reasonable manner, consistent with the particular problem considered, but, at this stage, no particular forms will be assigned to these functions.

By applying the kinematic relations of three-dimensional elasticity to the displacement approximation, one obtains the following approximate strain field:

$$\begin{aligned} \varepsilon_x &= e_x^c + z\, k_x^c + \varphi_1(z)\, k_x^a, & \gamma_{xz} &= \varphi_1'(z)\, e_{xz}^a + \psi(z)\, k_{xz}^a, \\ \varepsilon_y &= e_y^c + z\, k_y^c + \varphi_2(z)\, k_y^a, & \gamma_{yz} &= \varphi_2'(z)\, e_{yz}^a + \psi(z)\, k_{yz}^a, \\ \varepsilon_z &= \psi'(z)\, e_z^a, & \gamma_{xy} &= e_{xy}^c + z\, k_{xy}^c + \varphi_1(z)\, k_{xy}^a + \varphi_2(z)\, k_{yx}^a, \end{aligned} \tag{7}$$

where a prime denotes ordinary differentiation with respect to z, and

$$\begin{aligned} & e_x^c = u_{0,x}, && && k_x^a = u_{1,x}, \\ & e_y^c = v_{0,y}, && && k_y^a = v_{1,y}, \\ & e_{xy}^c = u_{0,y} + v_{0,x}, && e_z^a = w_1, && k_{xy}^a = u_{1,y}, \\ & && e_{xz}^a = u_1, && k_{yx}^a = v_{1,x}, \\ & k_x^c = -w_{0,xx}, && e_{yz}^a = v_1, && k_{xz}^a = w_{1,x}, \\ & k_y^c = -w_{0,yy}, && && k_{yz}^a = w_{1,y}. \\ & k_{xy}^c = -2w_{0,xy}, && && \end{aligned} \tag{8}$$

On the basis of strain energy and variational considerations [9–11], the force and moment resultants of the theory are defined as follows:

$$\begin{aligned} \left(N_x^c,\ N_y^c,\ N_{xy}^c\right) &= \int_{-h/2}^{h/2} (\sigma_x,\ \sigma_y,\ \tau_{xy})\, \mathrm{d}z, \\ \left(M_x^c,\ M_y^c,\ M_{xy}^c\right) &= \int_{-h/2}^{h/2} (\sigma_x,\ \sigma_y,\ \tau_{xy})\, z\, \mathrm{d}z; \\ \left(N_z^a,\ Q_x^a,\ Q_y^a\right) &= \int_{-h/2}^{h/2} \left(\sigma_z\, \psi'(z),\ \tau_{xz}\, \varphi_1'(z),\ \tau_{yz}\, \varphi_2'(z)\right)\, \mathrm{d}z, \\ \left(M_x^a,\ M_y^a,\ M_{xy}^a,\ M_{yx}^a\right) &= \int_{-h/2}^{h/2} \left(\sigma_x\, \varphi_1(z),\ \sigma_y\, \varphi_2(z),\ \tau_{xy}\, \varphi_1(z),\ \tau_{xy}\, \varphi_2(z)\right)\, \mathrm{d}z, \\ \left(P_x^a,\ P_y^a\right) &= \int_{-h/2}^{h/2} \left(\tau_{xz}\, \psi(z), \tau_{yz}\psi(z)\right)\, \mathrm{d}z. \end{aligned} \tag{9}$$

Here, quantities denoted with a superscript $^{(c)}$ are those met in CPT, while resultants denoted with a superscript $^{(a)}$ are predominantly associated with stress responses due to transverse shear and transverse normal deformation.

The six equations of equilibrium of the theory are obtained either variationally or with appropriate integration of the equilibrium equations (2) of three-dimensional elasticity through the plate thickness and, for the class of stationary thermo-elasticity problems considered, are as follows:

$$\begin{aligned} & N_{x,x}^c + N_{xy,y}^c = 0, \quad N_{xy,x}^c + N_{y,y}^c = 0, \quad M_{x,xx}^c + 2M_{xy,xy}^c + M_{y,yy}^c = q(x,y), \\ & M_{x,x}^a + M_{xy,y}^a - Q_x^a = 0, \quad M_{yx,x}^a + M_{y,y}^a - Q_y^a = 0, \quad P_{x,x}^a + P_{y,y}^a - N_z^a = \psi(h/2)\, q(x,y), \end{aligned} \tag{10}$$

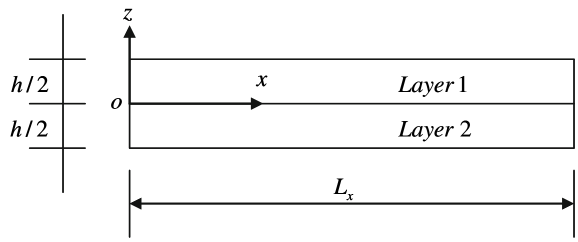


Fig. 2 Two-layered laminated beam geometry and coordinate system

where, in accordance with (1), $q(x, y)$ represents the form of some mechanical pressure loading which is possibly applied on the plate lateral planes.

Appropriate, 2D constitutive equations are obtained by inserting (3) and (4) into the definitions (9) of force and moment resultants and, then, by performing the denoted through-thickness integrations. These are similar in form to those presented in [10,11] and for the sake of brevity are not presented here. Due to the assumed through-thickness inhomogeneity, they contain a number of additional coupling and bending rigidities that complement their conventional counterparts met in the laminated-plate version of the CPT. Due, however, to the appearance in (2) of the absolute temperature field, T, they further contain a certain number of thermal-expansion rigidities [15], whose form and nature will become evident in the particular example application employed in the next section. Upon inserting those constitutive equations into (10), the latter set of six simultaneous PDEs is finally expressed in terms of the six main unknowns, u_0, v_0, w_0 u_1, v_1, and w_1 of the `G6DOFPT`.

4 Application: stationary thermo-elastic stress analysis of infinite strips

Consider the particular case of a flat plate of infinite extent in the y direction ($L_y = \infty$) and subjected to some kind of thermal and, possibly, mechanical lateral loading that are independent of both time and the y co-ordinate parameter. All plate cross-sections normal to the y-axis are then subjected to the same deformation pattern. Alternatively, each one of these cross-sections may be regarded as a prismatic beam having unit width, constant thickness h and axial length $L_x \equiv L$ (Fig. 2) and being elastically deformed in the xz-plane under the action of the considered time-independent thermal loading. The equations of the `G6DOFPT` presented in the preceding section are thus considerably simplified, due to the drop of all partial derivatives with respect to y. Hence, that simpler version of `G6DOFPT` may alternatively be considered as a generalised four-degree-of-freedom beam theory (`G4DOFBT`) that makes use of two shape functions only, namely $\varphi_1 \equiv \varphi$ and ψ ($\varphi_2 = 0$).

The formulation outlined in the preceding section leads therefore to four equations of stationary thermo-elastic equilibrium which may be finally expressed in terms of the four main unknown functions (degrees of freedom) involved as follows:

$$\begin{aligned}
&A^c_{11}u_{0,xx} - B^a_{11}w_{0,xxx} + B^a_{11}u_{1,xx} + B^b_{13}w_{1,x} = E^T_1,\\
&B^c_{11}u_{0,xxx} - D^c_{11}w_{0,xxxx} + D^a_{11}u_{1,xxx} + D^b_{13}w_{1,xx} = -q(x) + E^T_2,\\
&B^a_{11}u_{0,xx} - D^a_{11}w_{0,xxx} + D^{aa}_{11}u_{1,xx} - A^{aa}_{55}u_1 + (D^{ab}_{13} - A^{ab}_{55})w_{1,x} = E^T_3,\\
&-B^b_{13}u_{0,x} + D^b_{13}w_{0,xx} - (D^{ab}_{13} - A^{ab}_{55})u_{1,x} + A^{bb}_{55}w_{1,xx} - D^{bb}_{33}w_1 = \psi(h_{N+1})q(x) + E^T_4,
\end{aligned} \tag{11}$$

with the appearing mechanical and thermal expansion rigidities given according to,

$$\left(A^c_{11}, B^c_{11}, D^c_{11}, B^a_{11}, D^a_{11}, D^{aa}_{11}\right) = \int_{-h/2}^{h/2} C^{(r)}_{11}\left(1, z, z^2, \varphi, z\varphi, \varphi^2\right)\, \mathrm{d}z,$$

$$\left(B^b_{13}, D^b_{13}, D^{ab}_{13}\right) = \int_{-h/2}^{h/2} C^{(r)}_{13}\left(\psi', z\psi', \varphi\psi'\right)\, \mathrm{d}z,$$

$$\left(A_{55}^{aa}, A_{55}^{ab}, A_{55}^{bb}\right) = \int_{-h/2}^{h/2} C_{55}^{(r)} \left(\left(\varphi'\right)^2, \varphi'\psi, \psi^2\right) \, \mathrm{d}z, \quad D_{33}^{bb} = \int_{-h/2}^{h/2} C_{33}^{(r)} \left(\psi'\right)^2 \, \mathrm{d}z,$$

$$E_1^T = \int_{-h/2}^{h/2} (\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)}) \, (\Delta T)_{,x} \, \mathrm{d}z, \quad E_2^T = \int_{-h/2}^{h/2} (\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)}) \, (\Delta T)_{,xx} \, z \, \mathrm{d}z,$$

$$E_3^T = -\int_{-h/2}^{h/2} (\alpha_x^{(r)} C_{13}^{(r)} + \alpha_z^{(r)} C_{33}^{(r)}) \, \Delta T \, \psi' \mathrm{d}z, \quad E_4^T = -\int_{-h/2}^{h/2} (\alpha_x^{(r)} C_{13}^{(r)} + \alpha_z^{(r)} C_{33}^{(r)}) \, \Delta T \, \psi' \mathrm{d}z. \tag{12}$$

Here, a superscript T identifies rigidities that are mainly associated with thermal expansion, while $\psi(h_{N+1})q(x)$ is the contribution related to the aforementioned, possibly applied lateral mechanical loading. As already mentioned, the appearing thermal-gradient field ΔT should satisfy the heat-conduction equation.

Equations (11) form a tenth-order set of four simultaneous, inhomogeneous ordinary differential equations, thus requiring specification of five boundary conditions at each of the two strip edges (or beam ends). Explicit forms of all relevant sets of variationally consistent boundary conditions, applicable to the edges $x = 0,\ L$, may be found in [10,12]. For later use the following two sets of boundary conditions are only quoted here:

$$\text{simple support: } N_x^c = w = M_x^c = M_x^a = w_1 = 0, \tag{13}$$

$$\text{clamped edge: } u_0 = w_0 = w_{0,x} = u_1 = w_1 = 0. \tag{14}$$

Upon assuming, however, that the elastic moduli and the coefficients of linear thermal expansion do not depend on x, one may determine the general solution of (11) analytically, provided that the appropriate form of ΔT is also specified analytically and, hence, allows for the non-zero right-hand sides of (11) to be expressed in terms of elementary mathematical functions.

In the absence of lateral mechanical loading ($q(x) = 0$) and for the particular case of a non-uniform temperature field of the form:

$$\Delta T(x,z) = (T_0 + T_1 z)\sin(p_m x), \quad p_m = m\pi/L, \ \ (m = 1,\ 2, \ldots) \tag{15}$$

the general solution of the set of equations (12) is given in the Appendix. Expression (15) may be considered as a sine-Fourier-series harmonic of any relevant temperature field that satisfies the heat-conduction equation and, here, it is employed in connection with the particular example application considered later in Sect. 6. It should be emphasised that the general solution presented in the Appendix holds true regardless of the particular choice of the shape functions $\varphi(z)$ and $\psi(z)$. The simplest possible choices for these functions are $\psi(z) = 0$ and $\varphi(z) = z$ or $\varphi(z) = z(1 - 4z^2/3h^2)$, resulting in solutions equivalent to those obtained with the use of the Timoshenko [33] or the Bickford [34] beam theory, respectively. The most accurate set of these functions, determined instead in the next section, furnishes the present beam theory with the ability to provide identical results to those obtained by solving exactly the equations of anisotropic plane-strain elasticity for simply supported end boundaries.

5 Determination of a most accurate set of shape functions

For the class of strip problems introduced in the preceding section, a most accurate set of shape functions $\varphi(z)$ and $\psi(z)$ can be determined by making use of the plane-strain version of elasticity equilibrium equations, namely,

$$\sigma_{x,x} + \tau_{xz,z} = 0, \quad \tau_{xz,x} + \sigma_{z,z} = 0. \tag{16}$$

The plane-strain counterpart of the displacement expansion (6) is next used in the constitutive equations (3) which are introduced into Eq. 16. The following displacement pattern of a simply supported strip (or beam):

$$\begin{aligned} u_0 &= A_0 \cos p_m x, \quad u_1 = B_0 \cos p_m x, \\ w_0 &= C_0 \sin p_m x, \quad w_1 = D_0 \sin p_m x, \end{aligned} \tag{17}$$

converts the latter equations into two second-order simultaneous ODEs for the two unknown functions $\varphi(z)$ and $\psi(z)$ (see, for instance, [10,12,15] for further details).

The general solution of the latter set of ODEs may be expressed as follows [15]:

$$\left\{ \begin{matrix} B_0\varphi(z) \\ D_0\psi(z) \end{matrix} \right\} = \left\{ \begin{matrix} \Phi(z) \\ \Psi(z) \end{matrix} \right\} + \left\{ \begin{matrix} p_m \\ 0 \end{matrix} \right\} C_0 z - \left\{ \begin{matrix} A_0 \\ C_0 \end{matrix} \right\} + \left\{ \begin{matrix} f_1^{(r)} \\ f_2^{(r)} \end{matrix} \right\}, \tag{18}$$

where

$$\begin{aligned} f_1^{(r)} &= -(\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)})\,(T_0 + T_1 z)/(p_m C_{11}^{(r)}), \\ f_2^{(r)} &= \left[(C_{13}^{(r)} + C_{55}^{(r)})(\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)}) - C_{11}^{(r)}(\alpha_x^{(r)} C_{13}^{(r)} + \alpha_z^{(r)} C_{33}^{(r)}) \right] T_1/(p_m^2 C_{11}^{(r)} C_{55}^{(r)}), \end{aligned} \tag{19}$$

and the superscript (r), denoting the rth layer of the laminate considered, suggests that elastic moduli and coefficients of linear thermal expansion may differ from layer to layer. Moreover, $\Phi(z)$ and $\Psi(z)$ in the right-hand side of (18) represent the complementary solution of the aforementioned 4th-order set of ordinary differential equations. In the notation of the present paper, these functions are identical to their exact elasticity-solution counterparts determined in Sect. 2; they represent the exact, plane-strain-type thermo-elasticity solution of the strip problem proposed in the preceding section (see also [15,35]). The constants A_0, B_0, C_0 and D_0, also appearing in (17), may finally be determined by requiring of $\varphi(z)$ and $\psi(z)$ to satisfy the constraints,

$$\varphi(0) = \psi(0) = 0, \quad \left.\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right|_{z=0} = \left.\frac{\mathrm{d}\psi}{\mathrm{d}z}\right|_{z=0} = 1, \tag{20}$$

so that u_0 and w_0 represent displacements while u_1 and w_1 represent transverse strains that act on the strip middle plane (see also Sect. 3 and, for more details, [10,12,15]). As has already been mentioned in Sect. 3, although these functions may look analytically complicated, particularly if compared with their relevant simplest possible form employed in [33,34], they can easily be stored in a computer's memory and, hence, inserted in (12), where the denoted integrations are usually performed numerically.

It is worth noting in this context that, unlike their aforementioned simple counterparts [33,34] Φ and Ψ are both exponential functions of the transverse co-ordinate, z, with the exponents dependent on the material and the geometrical properties of the strip considered. Under these considerations, the most important feature of the merging of these shape functions with the "plane-strain"-type version of the 2D thermo-elastic plane model developed in Sect. 3 is that, for simply supported laminates, it yields identical displacement, strain and stress distributions with the corresponding, exact "plane-strain"-type thermo-elasticity solution [35]. It is then reasonable to expect that, for other sets of end conditions, solutions and therefore detailed response thermo-elastic characteristics of the strip are accurate, at least away of the laminate ends.

Nevertheless, a predictor–corrector method has also been employed for validation of the numerical results obtained for boundary conditions other than simply supported ones. Accordingly, the approximate bending stress initially predicted with the outlined approach is inserted into the elasticity equations of equilibrium which, in a corrector phase, are integrated in the z-direction and, if necessary, provide improved transverse-stress distributions. In all of the pure mechanical-loading applications considered previously [14,15,36,37], a very close agreement was generally sought and observed between corresponding inter-laminar stress distributions obtained with the application of the predictor and corrector phases of the method. Such an observation is always regarded as clear evidence that, for different than simply supported boundaries, the outlined method predicts very accurate stress distributions, even without the use of the outlined corrector phase.

6 Numerical results and discussion

In the following example, a clamped–clamped, two-layered anti-symmetric cross-ply laminated ($0^0/90^0$) beam is considered, with fibres in the bottom layer aligned along the x-axis. The beam thickness is determined by the ratio $L/h = 10$. It is assumed that the thermal deformation of the beam is due to a non-uniform temperature field (15). The orthotropic material considered has the following elastic properties:

$$E_L/E_T = 25, \quad G_{TT}/E_T = 0.5, \quad \nu_{LT} = 0.5, \quad \nu_{TT} = 0.25, \tag{21}$$

and the coefficients of thermal expansion are such that, $\alpha_L/\alpha_T = 0.1$ and 1.0 for the top and bottom layers, respectively, with the subscripts L and T denoting properties associated with the longitudinal and the transverse fibre direction, respectively.

In order to distinguish qualitatively between stress distributions induced by the uniform, $T_0 \sin(m\pi/L)$, and the linear along the z, $T_1 z \sin(m\pi/L)$, part of the temperature gradient, corresponding numerical results are depicted separately. All the numerical results illustrated in what follows are presented by means of the following nondimensional parameters:

(a) For the case of through-thickness uniform temperature variation ($T_1 = 0$),

$$\overline{U} = \frac{U}{\alpha_L T_0 L}, \quad \overline{W} = \frac{W}{\alpha_L T_0 L}, \quad \overline{\sigma_x} = \frac{\sigma_x}{E_T \alpha_L T_0}, \quad \overline{\sigma_z} = \frac{\sigma_z}{E_T \alpha_L T_0}, \quad \overline{\tau_{xz}} = \frac{\tau_{xz}}{E_T \alpha_L T_0}, \tag{22}$$

(b) For the case of through-thickness linear temperature variation with $T_0 = 0$,

$$\overline{U} = \frac{10\,U}{\alpha_L T_1 L^2}, \quad \overline{W} = \frac{10\,W}{\alpha_L T_1 L^2}, \quad \overline{\sigma_x} = \frac{10\,\sigma_x}{E_T \alpha_L T_1 L}, \quad \overline{\sigma_z} = \frac{10\,\sigma_z}{E_T \alpha_L T_1 L}, \quad \overline{\tau_{xz}} = \frac{10\,\tau_{xz}}{E_T \alpha_L T_1 L}. \tag{23}$$

Also, due to the geometrical symmetry of the problem considered, all four figures depict results for the left-half of the beam only ($0 \le x/L \le 0.5$).

Accordingly, for the aforementioned case (a), Figs. 3 and 4 illustrate the through-thickness normalised bending-stress distributions obtained in the predictor phase of the method and the corresponding transverse-shear-stress distribution predicted with the application of the corrector phase, respectively. Figures 5 and 6 illustrate the corresponding stress distributions related to case (b). More detailed numerical results that are not presented here have shown that the through-thickness distributions of transverse shear stresses obtained with the application of the predictor phase of the method are still continuous at the layer interface, although this is mainly due to the specific lay-up considered. Most importantly, with the exception of a narrow band in the vicinity of the clamped ends, the relative difference of the shear-stress values obtained by means of the predictor and the corrector phases employed was always particularly small. It is worth mentioning in this connection that this difference never exceeded 1% in the range $0.2 \le x/L \le 0.8$, for either uniform or linearly varying temperature field. This remarkable observation verifies further the accuracy of the present model in predicting accurate shear-stress distributions even directly, namely without the use of the corrector phase. Some, slightly different, further numerical results are presented in [38] while additional example applications that confirm the outlined trends and conclusions are discussed in [15].

7 Closure

The extension of the relevant purely mechanical theory [11,12] outlined in this paper takes advantage of solutions available in three-dimensional thermo-elasticity theory for simply supported laminated components (e.g. [22,28,35]) and, hence, enables the resulting two-dimensional ESL thermo-elastic model to operate on the basis of a small number of unknown degrees of freedom. It is shown in this connection that, within the framework of uncoupled thermo-elasticity theory, and provided that the thermal loading

Fig. 3 Normalised in-plane bending-stress distributions in predictor phase induced by $T_0 \sin(p_1 x)$

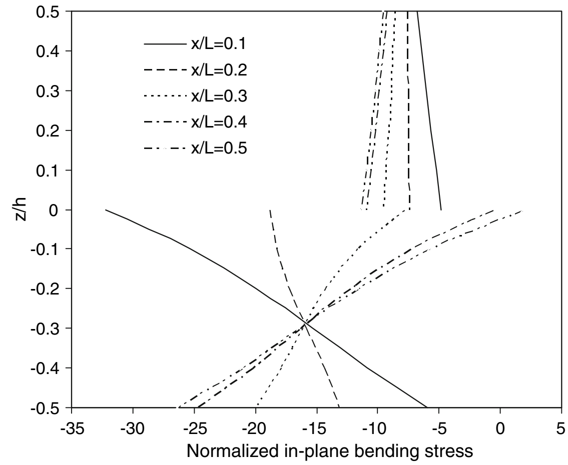


Fig. 4 Normalised transverse shear-stress distributions in corrector phase induced by $T_0 \sin(p_1 x)$

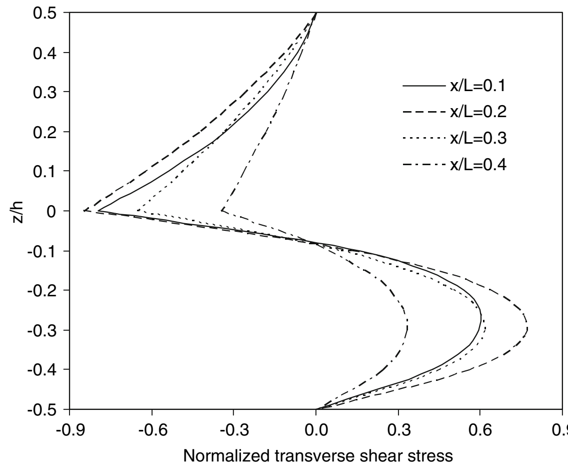
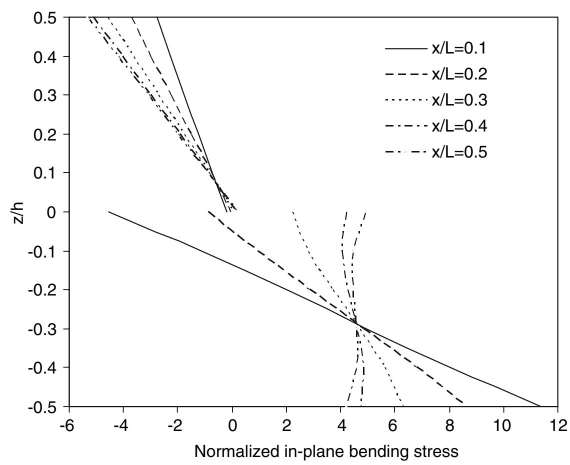


Fig. 5 Normalised in-plane bending-stress distributions in predictor phase induced by $T_1 z \sin(p_1 x)$



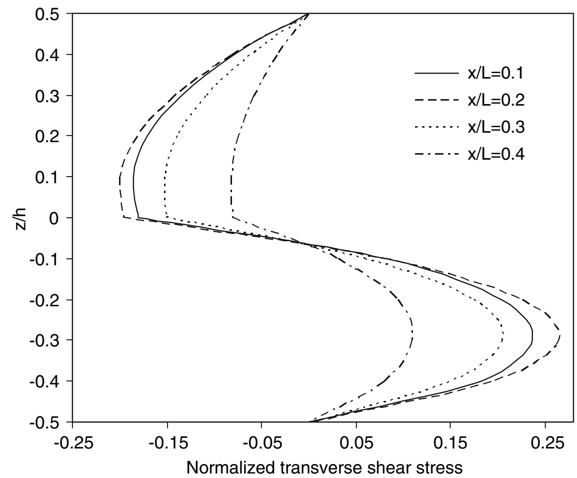


Fig. 6 Normalised transverse shear-stress distributions in corrector phase induced by $T_1\ z\ \sin(p_1 x)$

conforms to the requirements of the heat conduction equation, the number of the degrees of freedom involved (six) is kept as low as that employed in the original, purely mechanical counterpart [11,12] of the presented ESL theory. Hence, in possible applications that require use of coupled thermo-elasticity theory, no more than a small additional number of degrees of freedom may need to be incorporated into the model. These should account for an approximate through-thickness representation of the temperature field, T, which should be similar in form with the representation employed in (6) for the associated displacement components.

From an applied-mechanics viewpoint, the challenge faced by the presented, advanced ESL thermo-elastic model is related with the extent to which a possible thermo-elasticity generalisation of Saint Venant's principle accommodates accurately enough the prediction of thermo-elastic stresses away from the edges of a structural component which is not simply supported. The implemented predictor–corrector method (see also [10,15,36,37]) can still be found appealing and useful in this connection. Nevertheless, application of a relevant finite-element formulation [16] that was found particularly successful in cases of purely mechanical response of laminated composite components can also be extended further to account for relevant thermo-elasticity effects. Such a finite-element code is expected to be very efficient in cases involving possible sets of edge boundary conditions that could cause unexpected inaccuracy in the initial prediction of detailed, thermo-mechanical response characteristics of the composite component involved.

Relevant challenges include the possibility of extending successfully the method's applicability and usefulness in associated research areas of structural mechanics, such as eigenvibration analysis, linear stability and piezoelectric behaviour of composite and functionally graded structures and structural assemblies. In this connection, structural components with more complicated material, as well as geometrical configuration, should also be considered while the method's success should also be assessed in structural-analysis problems influenced by the presence of geometrical nonlinearity. Consideration of these, as well as other possible challenges, should be dealt with a welcome and positive attitude.

It is worth mentioning in this connection that micromechanics considerations and relevant effects that give rise to asymmetric theory of elasticity are expected to influence developments towards the generation of a new class of mathematical models for thin-walled structures. Resistance of fibres to bending and its consequences [39] may be referred to as an example that supports this expectation. In some detail, the relevant finite-deformations study presented in [39] has developed, as a particular case, a version of asymmetric linear elasticity theory that takes into consideration the effects of a fibre's bending stiffness. Though currently available in a form appropriate to the symmetries of transverse isotropy only, the latter version of three-dimensional elasticity theory can serve as a means for the incorporation of the mentioned micromechanics effects into accurate modelling of two- or one-dimensional thin-walled structures. Dealing, in particular, with the advanced elastic-beam modelling of the type outlined in the present paper, reference [39] has already provided a set of higher-order PDEs, the solution of which for simply supported components will yield an appropriate set of shape functions analogous to those described in (18) with the use of symmetric plane-strain elasticity. In the spirit outlined at the beginning of the Introduction, these challenges place the work and research output outlined in this paper no further than a point of possible reference for future developments in the subject of two-dimensional mathematical modelling of thin-walled structures and structural components.

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Appendix

The general solution of the system of ordinary differential equations (11) can be written in the following form, which is independent of the choice of the shape functions:

$$
\begin{aligned}
u_0 &= \frac{1}{F_1}\left\{\sum_{i=1}^{4}\frac{1}{\mu_i}\left[A_{55}^{aa}F_1F_5+\mu_i^2\left(F_3G_2-F_5G_1\right)\right]K_ie^{\mu_i x}+Q_3K_5x-Q_2\left(\frac{1}{2}K_6x^2+K_7x\right)+K_8\right\}\\
&\quad +A\cos p_m x,\\
w_0 &= \frac{1}{F_1}\left\{\sum_{i=1}^{4}\frac{1}{\mu_i^2}\left[A_{55}^{aa}F_1F_4+\mu_i^2\left(F_2G_2-F_4G_1\right)\right]K_ie^{\mu_i x}+\frac{1}{2}Q_2K_5x^2-Q_1\left(\frac{1}{6}K_6x^3+\frac{1}{2}K_7x^2\right)+K_9x\right.\\
&\quad \left.+K_{10}\right\}+C\sin p_m x,\\
u_1 &= \sum_{i=1}^{4}\mu_iG_2K_ie^{\mu_i x}+\frac{F_2G_4-F_4G_2}{A_{55}^{aa}F_1G_4}K_6+B\cos p_m x,\\
w_1 &= \sum_{i=1}^{4}\left(A_{55}^{aa}F_1-\mu_i^2G_1\right)K_ie^{\mu_i x}+\frac{F_5}{G_4}K_5-\frac{F_4}{G_4}\left(K_6x+K_7\right)+D\sin p_m x
\end{aligned}
\tag{A1}
$$

Here,

$$
\begin{aligned}
&F_1=A_{11}^cD_{11}^c-B_{11}^{c^2}, && G_1=B_{11}^aF_3-D_{11}^aF_2+D_{11}^{aa}F_1, && Q_1=A_{11}^c+\frac{F_4^2}{G_4},\\
&F_2=A_{11}^cD_{11}^a-B_{11}^cB_{11}^a, && G_2=B_{11}^aF_5-D_{11}^aF_4+\left(D_{13}^{ab}-A_{55}^{ab}\right)F_1, && Q_2=B_{11}^c+\frac{F_4F_5}{G_4},\\
&F_3=B_{11}^cD_{11}^a-D_{11}^cB_{11}^a, && G_3=B_{13}^bF_3-D_{13}^bF_2+\left(D_{13}^{ab}-A_{55}^{ab}\right)F_1, && \\
&F_4=A_{11}^cD_{13}^b-B_{11}^cB_{13}^b, && G_4=B_{13}^bF_5-D_{13}^bF_4+D_{33}^{bb}F_1, && Q_3=D_{11}^c+\frac{F_5^2}{G_4},\\
&F_5=B_{11}^cD_{13}^b-D_{11}^cB_{13}^b, && &&
\end{aligned}
\tag{A2}
$$

while, K_i $(i = 1, 2, \ldots, 10)$ are 10 arbitrary constants of integration and μ_i are the four roots of the following quartic algebraic equation:

$$
A_{55}^{bb}F_1G_1\mu^4-\left(G_1G_4-G_2G_3+A_{55}^{bb}A_{55}^{aa}F_1^2\right)\mu^2+A_{55}^{aa}F_1G_4=0. \tag{A3}
$$

The trigonometric terms appearing in the right-hand sides of (A1) represent the particular integral of the differential equations (11) and their coefficients are therefore determined by standard methods of differential calculus. As far as the present applications are concerned ($q = 0$), these are accordingly determined by inverting the following set of simultaneous algebraic equations:

$$\begin{bmatrix} -p_m^2 A_{11}^c & p_m^3 B_{11}^c & -p_m^2 B_{11}^a & p_m B_{13}^b \\ & -p_m^4 D_{11}^c & p_m^3 D_{11}^a & -p_m^2 D_{13}^b \\ & & -p_m^2 D_{11}^{aa} - A_{55}^{aa} & p_m \left(D_{13}^{ab} - A_{55}^{ab} \right) \\ \text{Symmetric} & & & -p_m^2 A_{55}^{bb} - D_{33}^{bb} \end{bmatrix} \begin{Bmatrix} A \\ C \\ B \\ D \end{Bmatrix} = \begin{Bmatrix} H_1^T \\ H_2^T \\ H_3^T \\ H_4^T \end{Bmatrix}, \tag{A4}$$

where,

$$H_1^T = \int_{-h/2}^{h/2} p_m(\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)})(T_0 + T_1 z)\, \mathrm{d}z, \quad H_2^T = -\int_{-h/2}^{h/2} p_m^2(\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z^{(r)} C_{13}^{(r)})(T_0 + T_1 z)\, z\, \mathrm{d}z,$$

$$H_3^T = \int_{-h/2}^{h/2} p_m(\alpha_x^{(r)} C_{11}^{(r)} + \alpha_z C_{13}^{(r)})(T_0 + T_1 z)\varphi \mathrm{d}z, \quad H_4^T = -\int_{-h/2}^{h/2} (\alpha_x^{(r)} C_{13}^{(r)} + \alpha_z^{(r)} C_{33}^{(r)})(T_0 + T_1 z)\, \psi' \mathrm{d}z. \tag{A5}$$

and the superscript (r) denotes the layer number.

The 10 arbitrary constants of integration K_i ($i = 1, 2, \ldots, 10$) appearing in (A1) are free to be determined by means of an appropriate set of boundary conditions imposed at the edges $x = 0$ and $x = L$ of the beam (e.g. (14)). This is achieved, by connecting expressions (A1) with the chosen set of edge boundary conditions and then solving the resulting 10×10 system of linear algebraic equations. In the particular case of a beam having both of its edges simply supported, the latter system returns $K_i = 0$ ($i = 1, 2, \ldots, 10$), and expressions (A1) are naturally reduced to the appropriate trigonometric form (19) that satisfies exactly the simple support boundary conditions at both ends, $x = 0,\ L$.

References

1. Meleshko VV (1997) Bending of an elastic rectangular clamped plate: Exact versus "engineering" solutions. J Elasticity 48:1–50
2. Ritz W (1909) Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik. J Reine Angew Math 13:1–61
3. Kirchhoff G (1850) Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. J Reine Angew Math 40:51–58
4. Reissner E (1985) Reflections on the theory of elastic plates. Appl Mech Rev 38:1453–1464; also in Proc 19th Midwestern mech. conf., Ohio State University, Columbus, 9–11 September, 1985, pp 32–58
5. Reissner E (1944) On the theory of bending of elastic plates. J Math Phys 23:184–191
6. Reissner E (1945) The effect of transverse shear deformations on the bending of elastic plates. J Appl Mech 12:A69–A77
7. Ball JM, James RD (2002) The scientific life of Clifford Ambrose Truesdell III. Arch Rational Mech Anal 161:1–26
8. Soldatos KP (2006) General solutions for the statics of anisotropic, transversely inhomogeneous elastic plates in terms of complex functions. Math Mech Solids 11:596–628
9. Soldatos KP (1995) Generalisation of variationally consistent plate theories on the basis of a vectorial formulation. J Sound Vibr 183:819–839
10. Soldatos KP (2003) Accurate stress analysis of laminated composite components. In: Altenbach H, Becker W (eds) Modern trends in composite laminates mechanics. CISM Courses and Lectures no. 448. Springer, pp. 69–132
11. Soldatos KP, Watson P (1997) Accurate stress analysis of laminated plates combining a two-dimensional theory with the exact three-dimensional solution for simply supported edges. Math Mech Solids 2:459–489
12. Soldatos KP, Watson P (1997) A general four-degrees-of-freedom theory suitable for accurate stress analysis of homogeneous and laminated composites beams. Int J Solids Struct 34:2857–2885
13. Shu X, Soldatos KP (2000) Cylindrical bending of angle-ply laminates subjected to different sets of edge boundary conditions. Int J Solids Struct 37:4289–4307
14. Soldatos KP, Liu SL (2001) On the generalised plane strain deformations of thick anisotropic composite laminated plates. Int J Solids Srtuct 38:479–482
15. Liu SL (2001) Stress analysis in laminated composites. PhD Thesis, Theoretical Mechanics Division, University of Nottingham
16. Soldatos KP, Shu X (2002) Improving the efficiency of FE formulations in laminated composites. Commun Num Meth Engng 18:605–613

17. Nowaki W (1975) Dynamic problems in thermoelasticity. Noordholff, Layden
18. Chandrasekharaiah DS (1986) Thermoelasticity with second sound: A review. Appl Mech Rev 39:355–376
19. Kaliski S (1965) Wave equations on thermoelasticity. Bull Acad Polon Sci Tech 13:253–260
20. Dhaliwal RS, Sherief HH (1980) Generalized thermoelasticity for anothotropic media. Quart Appl Math 38:1–8
21. Li X (1992) Generalized theory of thermoelasticity for an anisotropic medium. Int J Engng Sci 30:571–577
22. Soldatos KP, Ye JQ (1994) Three-dimensional static, thermodynamic and buckling analysis of homogeneous and laminated cylinders. Compos Struct 29:131–143
23. Soldatos KP, Hadjigeorgiou VP (1990) Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels. J Sound Vibr 137:369–384
24. Soldatos KP (1991) An iterative solution of a Bessel's equation based on torsional vibrations of orthotropic hollow cylinders. J Sound Vibr 151:149–152
25. Soldatos KP, Hawkes TD (1991) An iterative three-dimensional solution of the free vibration problem of homogeneous orthotropic hollow cylinders. In: Krätzig WB, Bruhns OT, Jessberger HL, Meskouris K, Niemann H-J, Schmid G, Stangenberg F, Kounadis AN, Schuëller GI (eds) Structural dynamics 2. Balkema, Rotterdam, The Netherlands pp 883–889
26. Hawkes TD, Soldatos KP (1992) Three-dimensional axisymmetric vibrations of orthotropic cross-ply laminated hollow cylinders. AIAA J 30:1089–1098
27. Ye JQ, Soldatos KP (1994) Three-dimensional vibrations of laminated cylinders and cylindrical panels with a symmetric or an antisymmetric cross-ply lay-up. Compos Engng 4:429–444
28. Soldatos KP, Ye JQ (1995) Stationary thermoelastic analysis of thick cross-ply laminated cylinders and cylindrical panels. Acta Mech 110:1–18
29. Soldatos KP, Ye JQ (1995) Three-dimensional buckling analysis of orthotropic cylinders subjected to thermo-mechanical loading. In: Parker DF, England AH (eds) IUTAM symposium on anisotropy, inhomogeneity and nonlinearity in solid mechanics (honouring Prof AJM Spencer on his retirement). Kluwer, Dordrecht pp 299–304
30. Shuvalov AL, Soldatos KP (2003) On the successive approximation method for three-dimensional analysis of radially inhomogeneous tubes with arbitrary cylindrical anisotropy. J Sound Vibr 259:233–239
31. Soldatos KP, Timarci T (1993) A unified formulation of laminated composite shear deformable five-degrees-of-freedom cylindrical shell theories. Compos Struct 25:165–171
32. Timarci T, Soldatos KP (2000) Vibrations of angle-ply laminated circular cylindrical shells subjected to different sets of edge boundary conditions. In: Hill JM, Selvadurai APS (eds) J Engng Math 37:211–230. Special issue in honour of Prof AJM Spencer
33. Timoshenko S (1922) On the transverse vibrations of bars of uniform cross-section. Philos Magaz 43:125–131
34. Bickford WB (1982) A consistent higher-order beam theory. Dev Theor Appl Mech 11:137–150
35. Murakami H (1993) Assessment of plate theories for treating the thermo-mechanical response of layered plates. Compos Engng 3:137–149
36. Liu S, Soldatos KP (2002) On the improvement of transverse stress distribution predictions in cross-ply laminated beams: advanced versus conventional beam modelling. Int J Mech Sci 44:287–304
37. Liu S, Soldatos KP (2003) Further assessment of a generalised plate model: stress analysis of angle-ply laminates. Int J Solids Struct 40:4125–4133. Special issue in honour of Prof AW Leissa.
38. Soldatos KP, Liu SL (2003) On accurate prediction of stress distributions in laminates subjected to thermo-mechanical loading. In: Librescu L, Marzocca P (eds) Thermal stresses '03; the 5th international congress on thermal stresses and related topics, Virginia Tech Inst & State Univ, Blacksburg 8–11 June 2003
39. Spencer AJM, Soldatos KP (2007) Finite deformations of fibre-reinforced elastic solids with fibre bending stiffness. Int J Non-lin Mech 42:355–368